

## MATHEMATICS CLUB HANDOUT SHEET Sept 21<sup>st</sup> 2013

### The Logistic Equation and Chaos

#### Introduction

You have probably heard of the great scientist and mathematician Sir Isaac Newton. One of Newton's greatest triumphs was to describe the motion of bodies, be they apples or planets, by means of relatively simple deterministic equations. Deterministic means, essentially, that all the parameters of the equations are known, with no uncertain elements. Newton's formulation essentially described a universe running like 'clockwork', with everything running according to his laws of motion in a regular *predictable* pattern. Newton's vision of an orderly world is still extremely important today, but what I want to show you today is how that vision has become somewhat distorted, ironically by the behaviour of those apparently simple, predictable, equations upon which it is based.

Imagine a number, let us say  $x_0$ , between zero and one. To get another number let us take  $x_0$ , multiply its difference from one  $1 - x_0$  and then multiply the result by any other positive number, which we will label  $a$ . The resulting number, say  $x_1$ , satisfies the equation

$$x_1 = ax_0(1 - x_0) = ax_0 - ax_0^2. \quad (1)$$

So for instance if  $x_0 = 0.5$  and  $a = 1.5$  then  $x_1 = 1.5 \times 0.5 \times 0.5 = 0.375$ . Equation (1) is a deterministic equation (the value of  $a$  is a fixed number) known as the (discrete) Logistic equation, and it actually plays a fundamental role in biology, in particular for predicting the behaviour of populations. To see why imagine  $x$  represents the population of some animal, for the sake of argument say elephants in a game reserve, measured in units of thousands. So  $x_0 = 0.5$  is equivalent to a population of  $1000 \times 0.5 = 500$  in total. Now how does the elephant population change to a new value  $x_1$ ? Well it probably depends on two factors. First it *increases* when new baby elephants are born, which depends on (or is proportional to) the numbers in the population, originally  $x_0$ . Hence the *positive* term  $ax_0$  that appears in (1). But elephants need considerable quantities of vegetation to survive, and if the population gets too large they will eat all the edible material (leaves, roots, even bark from trees) in the reserve. That is the elephants will consume all their food reserves or *logistics* and the population will *decrease* through starvation. Frequently this effect is found to depend on the level of the population squared, hence the *negative* term  $-ax_0^2$  appearing in (1), which acts as a balance restraining the population from ever getting too large.

We have seen that if  $a = 1.5$  and the initial elephant population is  $x_0 = 0.5$  (500 hundred) the corresponding logistic equation tells us it falls to  $x_1 = 0.375$  (375) in some specified time period (say a year for elephants). A legitimate question to ask is what happens to the population over the long term if we repeatedly (recursively) apply the logistic equation from an initial starting value? That is in years 2, 3... $n$  the population would given by

$$x_2 = ax_1(1 - x_1)$$

$$x_3 = ax_2(1 - x_2)$$

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$$x_n = ax_{n-1}(1 - x_{n-1})$$

and we would like to be able predict  $x_n$  from  $x_0$  as  $n$  gets large. With the help of the computer we shall attempt to do this.

### **Investigating the Logistic equation using the spreadsheet program Excel.**

We are going to use the spreadsheet program Excel to study the logistic equation (1). Log on to the computer in front of you (the Maths staff will instruct you how) and highlight the Microsoft Excel icon. You should see a typical spreadsheet made up of squares running from A, B, C, etc across the top and 1, 2, 3, etc downwards.

#### **Plotting Graphs**

Let's use the spreadsheet to establish what happens to our (elephant) population starting at  $x_0 = 0.5$  with  $a$  set to the value 1.5 as above.

In box A1 type  $a$  [Enter]

In box B1 type  $x_0$  [Enter]

In box C1 type **Year** [Enter]

In box D1 type **Population** [Enter]

These give some headings to the columns you are going to create.

Now in Box A2 type =**1.5** (the value of the constant  $a$ )

Now in Box B2 type =**0.5** (the value of  $x_0$  the initial population level)

Now in Box C2 type =**1** (represents year 1)

Now in Box D2 type =**A2\*B2\*(1-B2)** (the basic form of the logistic equation)

In D2 you should see the number 0.375 appearing almost magically. This is the value of  $x_1$  as calculated by the program, which we found earlier. To get some more values do the following.

In Box A3 type =**A2** (1.5 should appear again)

Now using the mouse highlight the A3 square. Carefully move the mouse to the lower right corner of the square. The large white cross should change to a small black one. Whilst pressing the left key of the mouse, drag right down the black cross to square A25. The number 1.5 should appear in all the boxes.

Go back to the top of your spreadsheet.

In box C3 type =**C2+1** (the number 2=1+1 should appear)

Now using the mouse highlight the C3 square. Carefully move the mouse to the lower right corner of the square. The large white cross should change to a small black one. Whilst pressing the left key of the mouse, drag right down the black cross to square C25. The numbers 3-24 should appear in the boxes C4-C25.

Go back to the top of your spreadsheet.

In box D3 type =**A3\*D2\*(1-D2)**

The number 0.3515625 should appear. This is the value of the population at year 2 after our second application (iteration) of the logistic equation.

Now using the mouse highlight the D3 square. Carefully move the mouse to the lower right corner of the square. The large white cross should change to a small black one. Whilst pressing the left key of the mouse, drag right down the black cross to square D24. There is no need to go any further at this stage. What you should see is a series of numbers filling the boxes D3 to D24 which gives the population values in those years. You should also see that quite quickly the population settles down to a *stable* value which remains unchanged as the years go by.

What is the value the population settles down to? .....

We can see things more clearly by plotting a graph.

Go back write to the top of the spreadsheet. Using the mouse highlight all the squares C1-C24-D1-D24. (Move the cursor to C1 and drag the mouse down and across to D24.)

Now point the cursor at the 'Insert' icon on the tool bar above the spreadsheet and click the mouse. On the middle of the icon bar you should see a section labelled 'Charts'. Move your mouse to the chart entitled 'Scatter' and click. Highlight the second graph in the middle row (on the right) by positioning the cursor and clicking (the mouse).

You should see a graph called 'Population' with the numbers 1-24 on the horizontal axis and the population value along the vertical axis. If you don't like the horizontal grid lines, you can get rid of them by moving the mouse to point at the lines and right clicking. Then highlight the 'delete' option and the lines will go. Enlarge the graph by highlighting the graphs frame and then dragging the mouse to expand the picture. It should show the population starting at the value 0.375 at year1 and falling to its fixed value as time goes on.

Are you happy with this value? If the population is stable that means from one year to the next it remains unchanged. That is if the population at year  $n$  is  $x_n$  then in year  $n + 1$  it is also  $x_n$ . This means from the logistic equation  $x_n$  satisfies

$$x_n = ax_n(1 - x_n). \tag{2}$$

Can you solve this equation? It is a quadratic in  $x_n$ , so it has two answers, but one should be obvious to you!

The two solutions to equation (2) are .....

Does the 'sensible' solution agree with your numerical result?

Let us try some new values. Delete the graph you have (just press the delete key). First let us try some new values for  $x_0$ . Go to the B2 square and type =**0.1**. What happens to the population now? Plot the graph as before. Does the final result change?

Try some other values of  $x_0$  between zero and one (or even see what happens when  $x_0$  equals zero or one!).

Now let's try changing  $a$ . Reset  $x_0 = 0.5$  in box B2 and go to box A2. Type in =**2.5** (sets  $a = 2.5$  is the new value). What happens now?

Does the population converge?

If it does, to what value?

Is it what you expected?

Try changing the  $x_0$  value and plotting a few graphs.

Now try changing the value of  $a$  first to 2.8, then 2.9 and then 2.95 by changing the value in box A2 as before.

What do you notice about the speed of the convergence of the population levels now?  
(If you want to look over more years highlight the four boxes A24, B24 C24 & D24 all together, move the cursor to the bottom right hand corner so it becomes a black cross and keeping the left hand button pressed drag the mouse down.)

Try changing  $x_0$  and plotting a few graphs as before.

Now reset  $x_0 = 0.5$  in box B2 and go to box A2. Type in =**3.2** (sets  $a = 3.2$  as the new value).

What happens?

What value is the population converging to now?

Plot out the graph as before. Are you surprised by this behaviour from this 'simple' equation?  
(Sir Isaac Newton would have been amazed!)

### Simple Iteration

The process we have been using to calculate the population levels is termed simple iteration. We take our initial value  $x_0$  and *map* it onto the value  $x_1$  by means of a rule or function of  $x$  that we can call  $g$ . That is

$$\begin{aligned}x_1 &= g(x_0) \\x_2 &= g(x_1) \\&'' \\&'' \\x_n &= g(x_{n-1})\end{aligned}$$

Here the function  $g$  stands for the right hand side of the logistic equation i.e.  $g(x) = ax(1 - x)$ . Now there is a mathematical theorem that tells us something about the convergence

properties of such a sequence  $x_1, x_2, \dots, x_n$  calculated from the recipe above. It says that such a sequence will converge onto a single value, which we can denote by  $\alpha$ , provided

$$|g'(\alpha)| \leq 1.$$

Here  $g'(\alpha)$  represents the gradient or slope (or for those of you who have done some calculus, the derivative) of the function  $g$  at the value  $x = \alpha$ . Now if  $g(x) = ax(1 - x)$  it turns out that  $g'(x) = a - 2ax$ . Now from our work above we know the convergence value  $\alpha$  in terms of  $a$ . So can you decide when the criterion  $|g'(\alpha)| \leq 1$  is satisfied for the logistic equation? (That is the modulus of  $g'(\alpha)$  is less than or equal to one; the modulus is when you take the positive value of a function even when it is negative, e.g.  $|1| = 1 = |-1|$ .)

The logistic equation converges to a fixed value provided the constant  $a$  .....

Use the spreadsheet to confirm your prediction about when the long term behaviour of the logistic equation as the constant  $a$  changes. What happens if  $a < 1$ ? Try different start values as well. For example what happens when  $a = 0.5$ , and  $x_0 = 0.5, 1.5, 2$  and  $2.5$ ?

### Increasing the value of $a$

Now go back to box A2 and try a new larger value for  $a = 3.5$  starting from  $x_0 = 0.5$ . Type in =3.5 in Box A2 and =0.5 in Box B2.

Plot a graph of the population as before and don't be afraid to expand the chart area right out on the right hand side to see what is going on.

What do you see now happening to the population level?

Between what values is it fluctuating?

Do these values depend on  $x_0$ ?

Is this behaviour different to the case when  $a = 3.2$ ?

Can you find the value of  $a$  (to within  $\pm 0.1$ ) when this change of behaviour occurs?

Finally change the value of  $a$  in box A2 to 3.8 by typing =3.8. What happens to the long term population  $x_n$  now? Try changing the value of  $x_0$  and see what happens. Is there any pattern? Can you find any constraints (maximum and minimum values) on  $x_n$ ?

Repeat the exercise setting  $a$  equal to 3.6, 3.9999 and 4.0.

## **Conclusions**

What I hope this little demonstration has shown is that starting from the relatively simple logistic equation one can, by progressively tweaking the value of the parameter  $a$ , move from orderly predictable progression in the population level, to a 'periodic' type behaviour with different characteristics, to a totally unpredictably chaotic behaviour of the population. These ideas lead to quite profound questions for biologists (and much interesting work for mathematicians!), because in nature populations are frequently observed fluctuating quite markedly. Often the main problem is to decide whether these fluctuations are actually due to external factors (e.g. a drought making vegetation scarce for the elephants), or whether they are caused by the inherent chaotic dynamics in the equation governing the population's development.

What I hope you will take away from the session is that just because we can describe the changes to a system by means of fixed equations, we can't assume it will behave in a nice orderly predictable way. Life is always more complicated than you think!

## **Further reading for those interested**

*Chaos* by James Gleick (Heinemann 1988) is a popular, accessible account which goes into much more detail than we have had time today.